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Abstract

This paper treats synchronization problem of chaotic systems from a novel point of view, by using a change of coordinates to transform chaotic systems into a common canonical form, for which the synchronization problem can be easily studied via reduced observer. Sufficient and necessary conditions are given and the proposed method is illustrated by the synchronization of Rössler chaotic system.

I. Introduction

Since chaotic system is quite sensitive to its parameters and initial conditions, it was thought to be excluded in practice for a long time. After [16] successfully synchronized two identical chaotic systems with different initial conditions, chaos synchronization has been intensively studied in various fields. Since the work of [14], unidirectional synchronization can be viewed as a special case of observer design problem. Many techniques arising from observation theory have been applied to the problem of synchronization, such as observers with linearizable dynamics [9], adaptive [7], generalized hamiltonian form based observers [18], algebraic method [19], [2], and so on [6], [5], [25].

However, most of proposed methods are based on the full order observer, which needs to estimate all states of the system, including the measurable states which are the outputs. In order to reduce the complexity of observer design, the so-called reduced observer was proposed, which needs only estimate unmeasurable states of the studied system. It was firstly introduced for linear systems to reduce the number of dynamical equations by estimating only the unmeasurable states. Then it is generalized for nonlinear dynamical systems by imposing the Lipschitz conditions for nonlinear terms [8], [22], [20] and invariant manifold [10]. It is known that the formalism to design nonlinear observer, including reduced observer, for general nonlinear systems is still missing, hence some researchers tried to solve this problem by introducing the concept of normal form, i.e. transform the nonlinear systems via a diffeomorphism into the normal form, which is easy to design an observer, and then estimate the original states by the inverse of the diffeomorphism. For this technique, reader can refer to [21], [17], [15], [12], [11], [3], [4], [24], [23] and the references therein. Moreover, in [14], authors pointed out that sufficient and necessary conditions to guarantee the existence of a reduced observer are not reported, thus need to be studied.

Inspired by the concept of normal form, for the synchronization problem of chaotic systems, this paper proposes a nonlinear canonical form which permits to designe a reduced observer. Sufficient and necessary geometrical conditions are also deduced to determine whether nonlinear systems can be transformed into such a form thought a diffeomorphism. After that a reduced observer is designed for the proposed canonical form.

The paper is organized as follows. In section II we give notations, the definition of reduced observer and also the problem statement. Then a nonlinear canonical form is proposed in Section III and Section IV gives necessary and sufficient geometric conditions which allow us to transform nonlinear systems into the proposed nonlinear canonical form. The corresponding reduced observer is studied in Section V and an illustrative example is presented in Section VI.

II. Notations, definitions and problem statement

Without loss of generality, we assume that the studied chaotic system can be written as follows:

$$\begin{cases} \dot{x}_1 = F_1(x) \\ \dot{x}_2 = F_2(x) \\ y = x_2 \end{cases} \tag{1}$$

where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^p$ are the states and $y \in \mathbb{R}^p$ is the measured output, $F_1 : \mathbb{R}^{r+p} \to \mathbb{R}^r$ and $F_2 : \mathbb{R}^{r+p} \to \mathbb{R}^p$ are smooth vector functions.

For (1), the following gives the definition of the reduced observer.

Definition 1: The dynamical system defined as follows:

$$\dot{\hat{x}}_1 = \widetilde{F}_1(\hat{x}_1, x_2)$$

where x_2 is the output of (1), is an asymptotically reduced observer for (1) if

$$\lim_{t\to\infty} \parallel \hat{x}_1(t) - x_1(t) \parallel = 0.$$

Moreover, it is said to be an exponentially reduced observer if

$$\parallel \hat{x}_1(t) - x_1(t) \parallel \leq a e^{-bt} \parallel \hat{x}_1(0) - x_1(0) \parallel$$

for $t > 0$, where a, b are both positive constants.

Let recall the famous result of reduced observer for linear systems. For the following system:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 \\ y = x_2 \end{cases} \tag{2}$$

where A_{ij} are decomposition matrices for $1 \leq i, j \leq 2$, then it is easy to see [13] the following dynamic is a reduced observer for (2):

$$\dot{\hat{x}}_1 = A_{11}\hat{x}_1 + A_{12}x_2 + K(A_{21}x_1 - A_{21}\hat{x}_1), \tag{3}$$

with $A_{21}x_1 = \dot{y} - A_{22}y$. The observation error $e = \hat{x}_1 - x_1$ is determined by the following linear system:

$$\dot{e} = (A_{11} - KA_{21})e$$

which can be stabilized by the choice of K, provided that the pair (A_{11}, A_{21}) is observable.

In fact, the reduced observer can be easily extended for a more generic linear system than (2) in the following form:

$$\begin{cases} \dot{x} = A_{11}x_1 + A_{12}\zeta + A_{13}\vartheta \\ \dot{\zeta} = A_{21}x_1 + A_{22}\zeta + A_{23}\vartheta \\ \dot{\vartheta} = \mu\left(x, \zeta, \vartheta\right) \\ y = \left(y_1^T, y_2^T\right)^T = (\zeta^T, \vartheta^T)^T \end{cases}$$

where $y_2 = \vartheta$ is another possible measured output, but a redundant one. With those two outputs, the reduced observer (3) can be still applied without taking into account the dynamic $\dot{\vartheta} = \mu\left(x, \zeta, \vartheta\right)$. Moreover this dynamic can be used to improve the robustness of the observation.

It is well-known that to design a reduced observer directly for nonlinear systems is still an open problem, however this paper tries to solve this problem from another point of view: we introduce a nonlinear canonical form which enables us to design a reduced observer, in such a way if nonlinear systems can be transformed into such a nonlinear canonical form, then we can design a reduced observer for nonlinear systems. Consequently this paper deals with the following problems:

- ***What is the canonical form which enables us to design a reduced observer?***
- ***What are the sufficient and necessary conditions to transform nonlinear systems (chaotic systems) to such a canonical form?***
- ***How to design a reduced observer for the proposed canonical form?***

III. Canonical form for reduced observer

In what follows, we will present first the nonlinear canonical form which will be studied in this paper, then the sufficient and necessary geometric conditions to transform a generic chaotic system to such an observable nonlinear canonical form will be analyzed. An exponentially reduced observer can be easily designed for the proposed nonlinear canonical form, which will be detailed in the next section.

A. Nonlinear canonical form

Assume that the studied chaotic system can be decomposed into the following form:

$$\dot{x} = F_1(x,\zeta,\vartheta) = f(x,\zeta,\vartheta) \tag{4}$$

$$\dot{\zeta} = F_{21}(x,\zeta,\vartheta) = \gamma_1(\zeta)H(x) + \gamma_2(\zeta,\vartheta) \tag{5}$$

$$\dot{\vartheta} = F_{22}(x,\zeta,\vartheta) = \varepsilon\left(x,\zeta,\vartheta\right) \tag{6}$$

$$y = \left(\zeta^T,\vartheta^T\right)^T \tag{7}$$

where $x \in \mathbb{R}^r$, $\zeta \in \mathbb{R}$, $\vartheta \in \mathbb{R}^p$, $y \in \mathbb{R}^{p+1}$, $f : \mathbb{R}^{r+p+1} \to \mathbb{R}^r$, $H \in \mathbb{R}$, $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}$. Moreover, it is assumed that (4-7) is observable and (4) is observable with respective to $H\left(x\right)$, which implies that the following 1-forms:

$$\theta_i = dL_f^{i-1}H \tag{8}$$

for $1 \leq i \leq r$ are independent.

Then, consider the following observable nonlinear canonical form:

$$\begin{cases} \dot{z} = Az + \beta(y_1)z_r + \rho(y) \\ \dot{\xi} = \alpha_1(y_1)z_r + \alpha_2(y) \\ \dot{\eta} = \mu\left(z, y\right) \\ y = \left(y_1^T, y_2^T\right)^T = \left(\xi^T, \eta^T\right)^T \end{cases} \tag{9}$$

where

$$\begin{aligned} z &= (z_1, \cdots, z_r)^T \in \mathbb{R}^r \\ \beta &= (\beta_1, \cdots, \beta_r)^T \in \mathbb{R}^r \\ \rho &= (\rho_1, \cdots, \rho_r)^T \in \mathbb{R}^r \\ \eta &= (\eta_1, \cdots, \eta_p)^T \in \mathbb{R}^p \\ \xi &\in \mathbb{R}, \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \end{aligned}$$

and

$$A = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

with $z_r = Cz$ and $C = (0, \cdots, 0, 1)$.

Remark 1: Since the nonlinear canonical form (9) is supposed to be observable, thus z_r in (9) can be observed from the output y_1, which implies $\alpha_1(y) \neq 0$.

In section V we will see that a reduced observer can be designed for the proposed canonical form (9). Thus the following sub section concerns the deduction of sufficient and necessary conditions in order to transform (4-7) into (9).

IV. Sufficient and necessary conditions for the existence of a diffeomorphism

This section is devoted to seeking necessary and sufficient geometric conditions which allow us to transform (4-7) into the proposed nonlinear canonical form (9).

For (4-7), denote

$$F = \left(f^T, F_2^T\right)^T, F_2 = \left(F_{21}^T, F_{22}^T\right)^T$$

and the decomposition of F_2 into F_{21} and F_{22} makes $\gamma_1(\zeta) \neq 0$.

After having defined θ_i for $1 \leq i \leq r$ in (8), then define the vector field τ_1 according to the following equations:

$$\begin{aligned} &\theta_j(\tau_1) = 0, \quad \text{for } 1 \leq j \leq r-1 \\ &\theta_r(\tau_1) = 1 \end{aligned} \tag{10}$$

By induction, we can define the family of vector fields τ_j as follows:

$$\tau_j = [\tau_{j-1}, f] \;\; \text{for} \;\; 2 \leq j \leq r.$$

As we will see, the existence of a diffeomorphism necessarily implies that all τ_i commutes with each other, i.e. $[\tau_i, \tau_j] = 0$ for $1 \leq i \leq r$ and $1 \leq j \leq r$. We suppose that this condition is satisfied, and assume that σ, $\nu_1, \cdots, \nu_p$ are the vector fields determined by the following equations:

1) $[\tau_i, \tau_j] = [\tau_i, \sigma] = [\tau_i, \nu_l] = [\sigma, \nu_l] = [\nu_l, \nu_s] = 0$ for $1 \leq i, j \leq r$ and $1 \leq l, s \leq p$;
2) $d\zeta(\sigma) = 1$ and $d\vartheta(\sigma) = 0$;
3) $d\zeta(\nu_j) = 0$ and $d\vartheta_i(\nu_j) = \delta_i^j$ for $1 \leq i, j \leq p$ where δ_i^j represents Kronecker delta, i.e. $\delta_i^k = 1$ if $i = k$, otherwise $\delta_i^k = 0$.

Set

$$\begin{aligned} \theta &= (\theta_1, \cdots, \theta_r, d\zeta, d\vartheta_1, \cdots, d\vartheta_p)^T \\ \tau &= (\tau_1, \cdots, \tau_r, \sigma, \nu_1, \cdots, \nu_p)^T \end{aligned}$$

and denote $\Lambda = \theta(\tau_i, \sigma, \nu_j)$ for $1 \leq i \leq r$ and $1 \leq j \leq p$ the evaluation of θ over τ. Due to the observability property, Λ is invertible, hence we can define the following multi 1-forms

$$\omega = \Lambda^{-1}\theta = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{11}$$

where $\omega_2 = (d\zeta, d\vartheta_1, \cdots, d\vartheta_p)^T$ and ω_1 is the rest of ω. Then we are ready to state our main result.

Theorem 1: There exists a diffeomorphism $(z^T, \xi^T, \eta^T) = \phi(x, \zeta, \vartheta)$ which transforms the dynamical system (4-7) into the nonlinear canonical form (9) if and only if the following conditions are satisfied:

1) $[\tau_i, \tau_j] = 0$, for $1 \leq i \leq r$ and $1 \leq j \leq r$;
2) $[\tau_r, \bar{F}] = V(y_1)$ is a vector field which only depends on y_1 modulo the sub space spanned by $\{\tau_i, \sigma\}$ for $1 \leq i \leq r$;

3) $[\tau_j, F_2] \in \ker \omega_1$ for $1 \leq j \leq r-1$.

■

Proof: **Necessity:** Indeed, if (4-7) can be transformed into (9) via the diffeomorphism $(z^T, \xi^T, \eta^T) = \phi(x, \zeta, \vartheta)$, then $\tau_i = \frac{\partial}{\partial z_i}$ for $1 \leq i \leq r$, $\sigma = \frac{\partial}{\partial \xi}$ and $\upsilon_l = \frac{\partial}{\partial \eta_l}$ for $1 \leq l \leq p$. And it is easy to check that all conditions of Theorem 1 are satisfied.

Sufficiency: Consider the multi 1-forms ω defined in (11), we have $\omega(\tau) = \mathbf{I}_{(r+p+1)\times(r+p+1)}$, which implies $\omega(\tau_i)$ for $1 \leq i \leq r$, $\omega(\sigma)$ and $\omega(\upsilon_l)$ for $1 \leq l \leq p$ are constant. Therefore,

$$\begin{aligned} d\omega(\tau_i, \tau_k) &= L_{\tau_i}\omega(\tau_k) - L_{\tau_k}\omega(\tau_i) - \omega([\tau_i, \tau_k]) \\ &= -\omega([\tau_i, \tau_k]) \end{aligned}$$

thus, we can calculate vector fields σ and $\upsilon_1, \cdots, \upsilon_p$, such that $\{\tau_i, \sigma, \upsilon_l\}$ forms a basis, satisfying the equations mentioned above. Following the same principle, we have

$$\begin{aligned} d\omega(\tau_i, \sigma_k) &= -\omega([\tau_i, \sigma_k]), \\ d\omega(\tau_i, \upsilon_l) &= -\omega([\tau_i, \upsilon_l]) \\ d\omega(\upsilon_l, \upsilon_t) &= -\omega([\upsilon_l, \upsilon_t]) \end{aligned}$$

Since ω is an isomorphism, this implies the equivalence between

$$[\tau_i, \tau_k] = [\tau_i, \sigma] = [\tau_i, \nu_l] = [\sigma, \nu_l] = [\nu_l, \nu_t] = 0$$

and $d\omega = 0$.

According to theorem of Poincaré [1], $d\omega = 0$ implies that there exists a local diffeomorphism $(z^T, \xi^T, \eta^T) = \phi(x, \zeta, \vartheta)$ such that $\omega = d\phi$. We note $\omega_i = d\phi_i$ for $1 \leq i \leq 2$.

Since condition (1) in Theorem 1 is satisfied and τ is a basis, it implies $\phi_*(\tau_i) = \frac{\partial}{\partial z_i}$, $\phi_*(\sigma) = \frac{\partial}{\partial \xi}$ and $\phi_*(\upsilon_l) = \frac{\partial}{\partial \eta_l}$ for $1 \leq i \leq r$ and $1 \leq l \leq p$.

Now let us clarify the affect of this transformation on $f(x, \zeta, \vartheta)$ defined in (4). By the diffeomorphism $\phi(x, \zeta, \vartheta)$, we have $\begin{pmatrix} \dot{z} \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \phi_*(F)$ where $F = \begin{pmatrix} f \\ F_2 \end{pmatrix}$. It is easy to see that

$$\omega \begin{pmatrix} f \\ F_2 \end{pmatrix} = \begin{pmatrix} \omega_1(f + F_2) \\ \omega_2(f + F_2) \end{pmatrix} = \begin{pmatrix} \omega_1(f) + \omega_1(F_2) \\ \omega_2(f) + \omega_2(F_2) \end{pmatrix}$$

Then, for $1 \leq i \leq r$, we get

$$\begin{aligned}
\frac{\partial\left(\phi_*\left(F\right)\right)}{\partial z_i} &= \begin{pmatrix} \left[\omega_1\left(\tau_i\right), \omega_1\left(f\right) + \omega_1\left(F_2\right)\right] \\ \left[\omega_2\left(\tau_i\right), \omega_2\left(f\right) + \omega_2\left(F_2\right)\right] \end{pmatrix} \\
&= \begin{pmatrix} \omega_1\left[\tau_i, f\right] + \omega_1\left[\tau_i, F_2\right] \\ \left[\omega_2\tau_i, \omega_2\left(f\right) + \omega_2\left(F_2\right)\right] \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial z_{i+1}} \\ \left[\omega_2\left(\tau_i\right), \omega_2\left(f\right) + \omega_2\left(F_2\right)\right] \end{pmatrix}
\end{aligned}$$

since condition (3) $[\tau_i, F_2] \in \ker \omega_1$ for $1 \leq i \leq r-1$ implies $\omega_1[\tau_i, F_2] = 0$.

By integrating we obtain: $\omega_1\left(F\right) = Az + \varrho(y, z_r)$.

Moreover, as we know

$$\frac{\partial}{\partial z_k} H \circ \phi^{-1} \quad = dH(\tau_k) = \theta_1(\tau_k)$$

then according to the definition of τ_1 in (10), we get

$$\frac{\partial}{\partial z_r} H \circ \phi^{-1} = 1$$

which implies (5) can be written as

$$\dot{\zeta} = \gamma_1(\zeta) z_r + \gamma_2(\zeta) \tag{12}$$

Hence, by setting $\omega_2 = d\phi_2$ where $\phi_2 = \mathbf{I}_{(p+1)\times(p+1)}$, then we get

$$\phi_*\left(F\right) = \begin{pmatrix} Az + \varrho(y, z_r) \\ \gamma_1(y_1) z_r + \gamma_2(y) \\ \mu\left(z, \xi, \eta\right) \end{pmatrix} \tag{13}$$

Finally, by the condition (2), we have

$$\begin{aligned}
\frac{\partial \phi_*(\bar{F})}{\partial z_r} &= \phi_*\left(\left[\tau_r, \bar{F}\right]\right) \\
&= \phi_*\left(\sum_{j=1}^r V_j(y_1)\tau_j + W(y_1)\sigma\right) \\
&= \sum_{j=1}^r V_j(y_1)\frac{\partial}{\partial z_j} + W(y_1)\frac{\partial}{\partial \xi}
\end{aligned}$$

which means $W(y_1) = \alpha_1(y_1)$ and $\varrho(y, z_r)$ in (12) can be decomposed as:

$$\varrho(y, z_r) = \beta(y_1) z_r + \rho(y)$$

Thus we proved that (4-7) can be transformed to (9). ∎

V. Reduced observer design for the canonical form

If Theorem 1 is satisfied, then (4-7) can be transformed into (9) by a diffeomorphism $(z^T, \xi^T, \eta^T) = \phi(x, \zeta, \vartheta)$. This section is devoted to designing a reduced observer for the deduced nonlinear canonical form. Once the states of the canonical form have been estimated, by the inverse of the diffeomorphism, we can then recover the states of original chaotic system in the form of (4-7).

First of all, for (9) if we can accurately measure y_1, y_2 and calculate $\dot{y}_1$, this allows us to define a "new" output Y being a function of known output y and the derivative of y_1 in (9):

$$Y = \alpha_1^{-1}(y_1)\left(\dot{y}_1 - \alpha_2(y)\right) \tag{14}$$

Then we have the following preliminary result.

Proposition 1: The following dynamical system:

$$\dot{\hat{z}} = A\hat{z} + \beta(y_1)C\hat{z} + \rho(y) - K(y_1)(Y - C\hat{z}) \tag{15}$$

where $K(y_1) = -\beta(y_1) + \kappa$ and Y is defined in (14), is an exponentially reduced observer for (9), if the chosen κ makes $(A + \kappa C)$ Hurwitz.

Proof: Let $e = \hat{z} - z$ be the estimation error. Since $z_r = Cz$, then we can easily derive the dynamic of observation error from (9) and (15) as follows

$$\dot{e} = [A + (\beta(y_1) + K(y_1))C]e \tag{16}$$

Since the gain matrix $K(y_1)$ can be freely chosen, hence without loss of generality we set

$$K(y_1) = -\beta(y_1) + \kappa$$

which makes (16) become

$$\dot{e} = (A + \kappa C)e. \tag{17}$$

Consequently, if κ is chosen in such a way that matrix $(A+\kappa C)$ is Hurwitz, then the exponential convergence of $\hat{z}$ to z can be guaranteed. ■

Let us remark that the proposed reduced observer (15) is based on the "new" output Y defined in (14), which clearly shows that the derivative of the real output y_1 should be calculated according to (14). However, it is well-known that the derivative of noisy signal should be avoided if possible in practice, since derivative operation will amplify the influence of noise. Hence, in order to overcome this problem, we introduced an algebraic estimator as follows:

$$\varsigma = \hat{z} + \Gamma(y_1) \tag{18}$$

where $\Gamma(y_1) = \int K(y_1)\alpha_1^{-1}(y_1)dy_1$.

Remark 2: The term $\Gamma(y)$ always exists since $y_1 \in \mathbb{R}$ and $\alpha(y_1)$ is invertible, and it can be obtained by the following procedure:

1) being given A and C, determine κ such that $(A + \kappa C)$ is Hurwitz;
2) being given $\beta(y_1)$ and determined κ, calculate $K(y_1) = -\beta(y_1) + \kappa$;
3) being given the invertible $\alpha(y_1)$ and calculated $K(y_1)$, compute the integral $\Gamma(y_1) = \int K(y_1)\alpha_1^{-1}(y_1)dy_1$.

Remark 3: It should be noted that $\Gamma(y_1)$ defined above can be considered as a signal of filtered y_1, and thus limit the influence of noise on y_1.

Inserting Y defined in (14) into (15), it leads:

$$\begin{aligned} \dot{\hat{z}} + K(y_1)\alpha_1^{-1}(y_1)\dot{y}_1 = \ & (A + \beta(y_1)C + K(y_1)C)\hat{z} \\ & + \rho(y) + K(y_1)\alpha_1^{-1}(y_1)\alpha_2(y) \end{aligned} \tag{19}$$

In order to avoid the derivative of y_1, we take the new variable ς into account, then a more practical reduced observer can be derived from (19) as follows

$$\begin{aligned} \dot{\varsigma} &= (A + \beta(y_1)C + K(y_1)C)\left(\varsigma - \Gamma(y_1)\right) \\ &\quad + \rho(y) + K(y_1)\alpha_1^{-1}(y_1)\alpha_2(y_1) \\ &= (A + \kappa C)\varsigma + \rho(y) - (A + \kappa C)\Gamma(y_1) \\ &\quad + K(y_1)\alpha_1^{-1}(y_1)\alpha_2(y) \end{aligned} \tag{20}$$

with $\Gamma(y_1)$ defined in (18).

The proof of the convergence of $\varsigma - \Gamma(y_1)$ in (20) to z of the canonical form (9) is evident, and then one can estimate x of (4-7) by ϕ^{-1}. The next section gives an example to illustrate the feasibility of the proposed method.

VI. Illustrative example

In order to highlight the proposed method in this paper, let consider the well-known Rössler chaotic system as follows:

$$\begin{cases} \dot{x}_1 = x_2 + ax_1 \\ \dot{x}_2 = -x_1 - x_3 \\ \dot{x}_3 = c + x_3\left(x_2 - b\right) \\ y = x_3 \end{cases} \tag{21}$$

with $a = c = 0.2$, $b = 5.7$.

Fig. 1 depicts the chaotic attractor of Rössler system when $a = c = 0.2$, $b = 5.7$, from which we can see that x_3 is always positive, thus there is on observation singularities of x_1 and x_2.

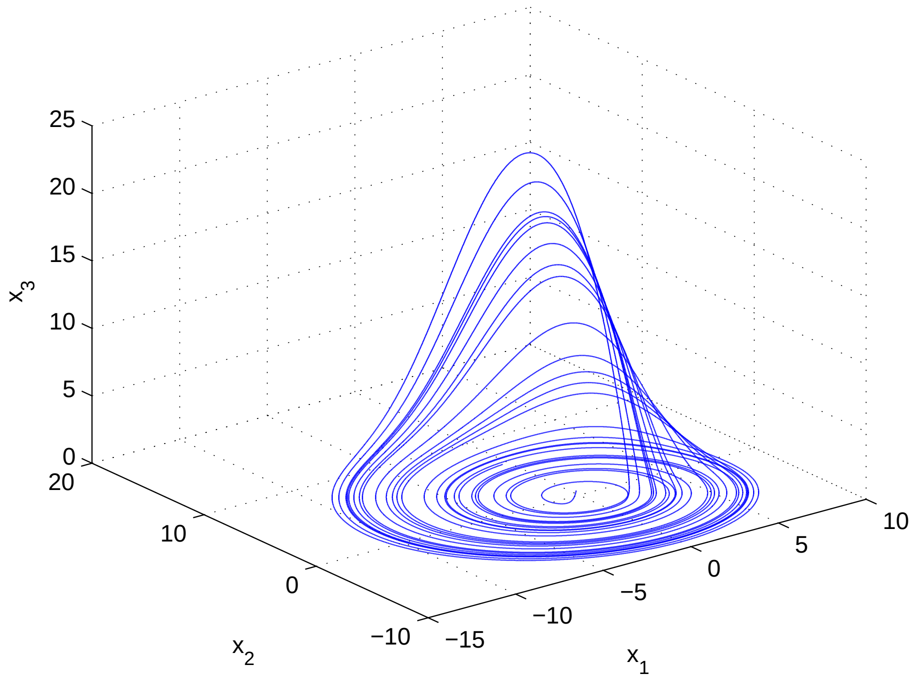

Fig. 1. Phase portrait of Rössler chaotic system with initial conditions $x_1(0) = -0.8$, $x_2(0) = -1$ and $x_3(0) = 1$.

By setting $x = (x_1, x_2)^T$ and $\zeta = x_3$, we can rewrite (21) into the following form:

$$\begin{cases} \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 + ax_1 \\ -x_1 - \zeta \end{pmatrix} \\ \dot{\zeta} = c + \zeta\left(x_2 - b\right) \\ y = \zeta \end{cases} \tag{22}$$

which is of the form (4-6) with

$$H\left(x\right) = x_2, f = \begin{pmatrix} x_2 + ax_1 \\ -x_1 - \zeta \end{pmatrix}$$
$$F_2 = c + \zeta\left(x_2 - b\right), \bar{F} = \left(f^T, F_2^T\right)^T$$

Then we can define the following 1-forms:

$$\theta_1 = dx_2, \theta_2 = -dx_1 - dx_3 \text{ and } d\zeta = dx_3$$

which yields

$$\tau_1 = -\frac{\partial}{\partial x_1}, \tau_2 = -a\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \text{ and } \sigma = \frac{\partial}{\partial x_3}$$

It is easy to check that $[\tau_1, \tau_2] = 0$ and $[\tau_i, \sigma] = 0$ for $1 \leq i \leq 2$. Moreover we have:

$$\begin{aligned} [\tau_2, \bar{F}] &= \left(1 - a^2\right)\frac{\partial}{\partial x_1} + a\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \\ &= -\tau_1 + a\tau_2 + x_3\sigma_1 \end{aligned}$$

In order to calculate the diffeomorphism, let compute:

$$\Lambda = \theta\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and it yields

$$\omega = \Lambda^{-1}\theta = \begin{pmatrix} -1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which gives $\omega_1 = (-x_1 - ax_2, x_2)^T$.

Then one can check that $\omega_1 [\tau_1, F_2] = 0$, implying $[\tau_1, F_2] \in \ker \omega_1$. Thus all conditions of Theorem 1 are satisfied, and one deduces the following diffeomorphism:

$$\phi = (z_1, z_2, \xi)^T = (-x_1 - ax_2, x_2, \zeta)^T$$

which is a global change of coordinates and can transform (21) into

$$\begin{cases} \dot{z}_1 = -z_2 + a\xi, \dot{z}_2 = z_1 + az_2 - \xi \\ \dot{\xi} = z_2\xi + c - b\xi \\ y = \xi \end{cases} \tag{23}$$

which is in the proposed normal form (9) with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} -1 \\ a \end{pmatrix}, \rho = \begin{pmatrix} a\xi \\ -\xi \end{pmatrix}$$
$$C = (0, 1), \alpha_1 = y, \alpha_2 = c - by$$

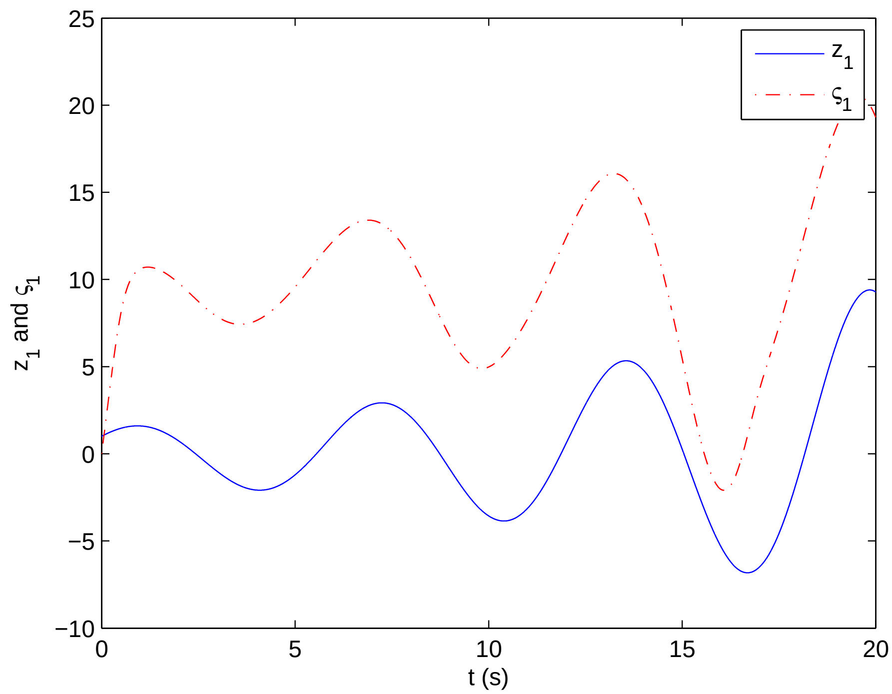

Fig. 2. z_1 and the estimate of ς_1 with initial conditions $z_1(0) = 1$ and $\varsigma_1(0) = 0$.

One can then design a reduced observer for (23). Being given A and C defined above, it is easy to find κ to make $(A + \kappa C)$ Hurwitz. In our simulation, we chose

$$\kappa = (-4, -4)^T$$

such that $(A + \kappa C)$ has two equal eigenvalues -2. With the above given β and the chosen κ, one obtains

$$K(y) = -\beta(y) + \kappa = \begin{pmatrix} -3 \\ -a-4 \end{pmatrix}$$

Then, according to the definition of $\Gamma(y)$, one gets

$$\Gamma(y) = \int K(y)\alpha^{-1}(y)dy = \int \begin{pmatrix} -3 \\ -a-4 \end{pmatrix} \frac{1}{y} dy = \begin{pmatrix} -3 \\ -a-4 \end{pmatrix} \ln|y|$$

As shown in Fig. 1, y is always positive (if $y = 0$, then system (21) is not observable), thus

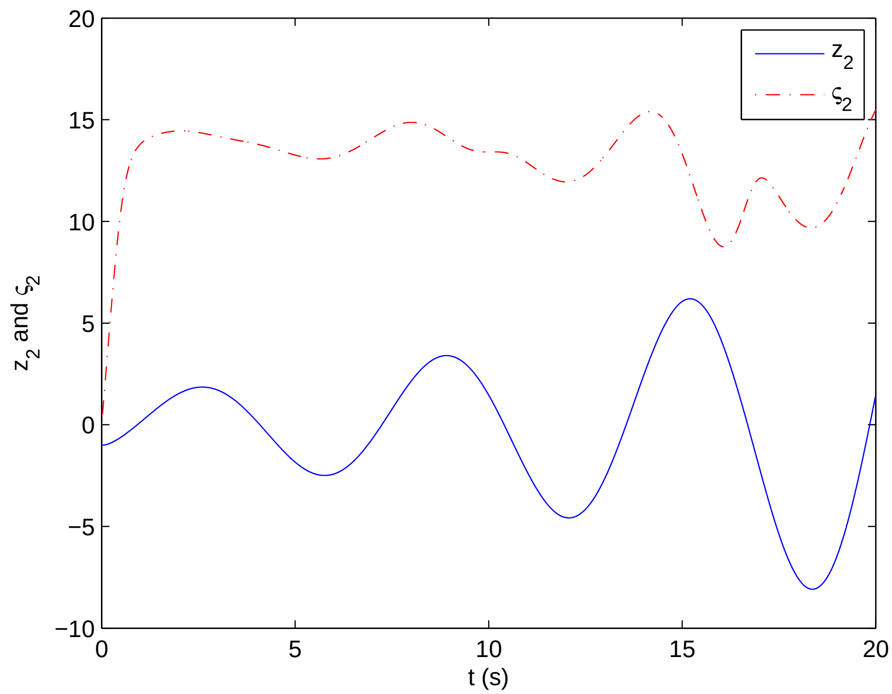

Fig. 3. z_2 and the estimate of ς_2 with initial conditions $z_2(0) = -1$ and $\varsigma_2(0) = 0$.

the deduced $\Gamma(y)$ can be simply written as

$$\Gamma(y) = \begin{pmatrix} -3 \\ -a-4 \end{pmatrix} \ln y$$

Thus, according to Proposition 1, the reduced observer for (23) can be designed as follows:

$$\begin{aligned} \dot{\hat{z}}_1 &= -\hat{z}_2 + a z_3 + 3\left(z_2 - \hat{z}_2\right) \\ \dot{\hat{z}}_2 &= \hat{z}_1 + a\hat{z}_2 - z_3 + (a+4)\left(z_2 - \hat{z}_2\right) \end{aligned}$$

where $z_2 = \left(\dot{y} + by - c\right)/y$, which explicitly depends on the derivative of y.

Since the derivative of the output will amplify the influence of noise, so in order to overcome this problem, let introduce the following algebraic estimator:

$$\varsigma = \hat{z} + \Gamma(y) = \hat{z} + \begin{pmatrix} -3 \\ -a-4 \end{pmatrix} \ln y$$

Then, after a straightforward calculation, one has the following reduced observer for (23) in the

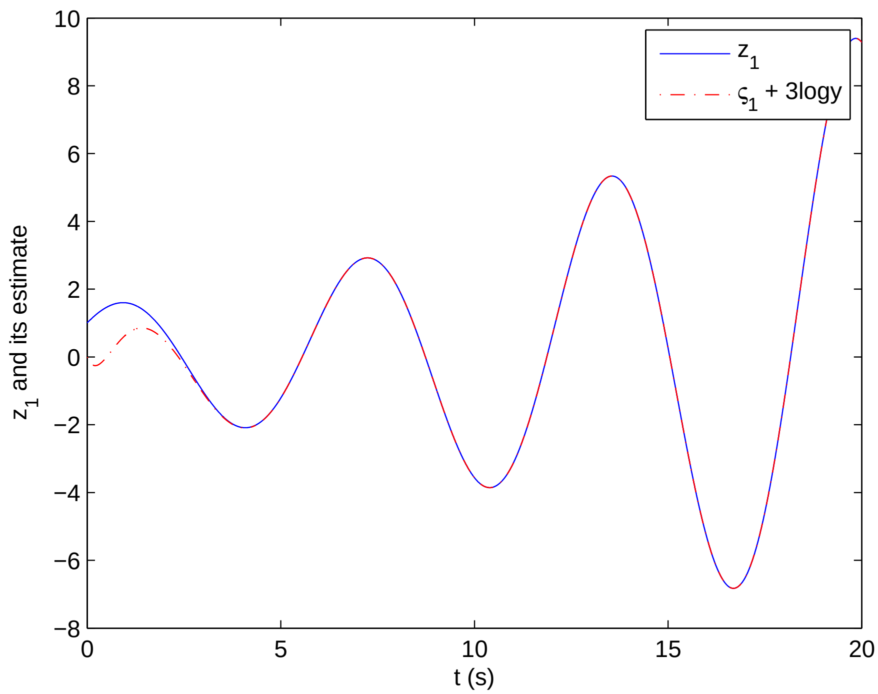

Fig. 4. z_1 and its estimate with initial conditions $z_1(0) = 1$ and $\hat{z}_1(0) = \varsigma_1(0) = 0$.

coordinate of ς:

$$\dot{\hat{\varsigma}}_1 = -4\hat{\varsigma}_2 + ay + (4a + 16)\ln y - 3\,(c - by)\,/y$$
$$\dot{\hat{\varsigma}}_2 = \hat{\varsigma}_1 - 4\hat{\varsigma}_2 - y + (4a + 13)\ln y - (a + 4)\,(c - by)\,/y$$

Fig. 2-7 are the simulations of the obtained reduced observer for (23), and it is shown that the deduced observer can perfectly estimate the unmeasurable states without estimating the output.

VII. Conclusion

The synchronization problem of chaotic systems via reduced observer is studied in this paper. We proposed a new canonical form which enables us to design a reduced observer. Sufficient and necessary conditions are given to determine whether a chaotic system can be transformed into such a form. Then a reduced observer is studied for the proposed canonical form, and the feasibility is highlighted by an illustrative example.

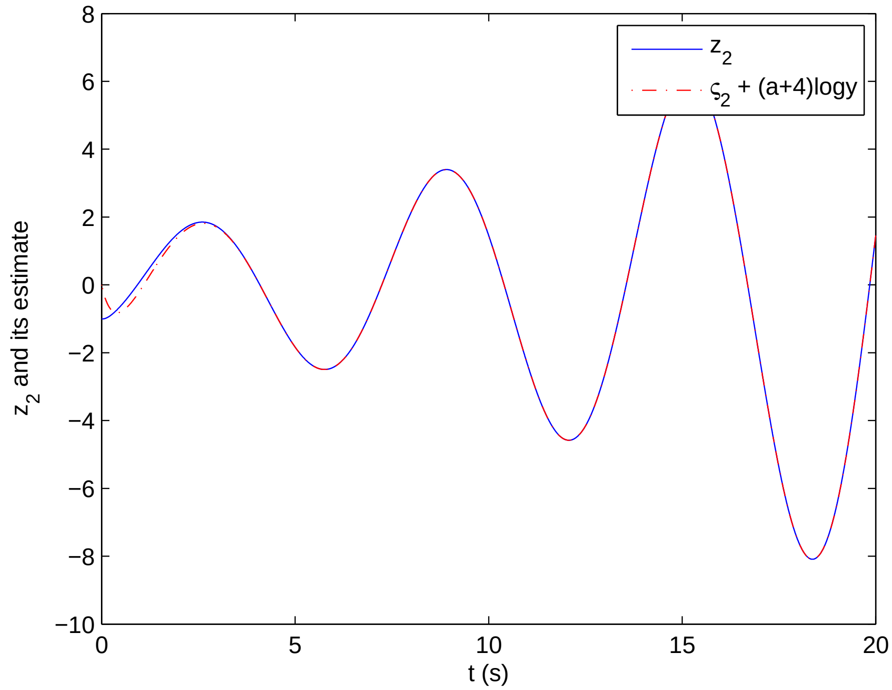

Fig. 5. z_2 and its estimate with initial conditions $z_2(0) = -1$ and $\hat{z}_2(0) = \varsigma_2(0) = 0$.

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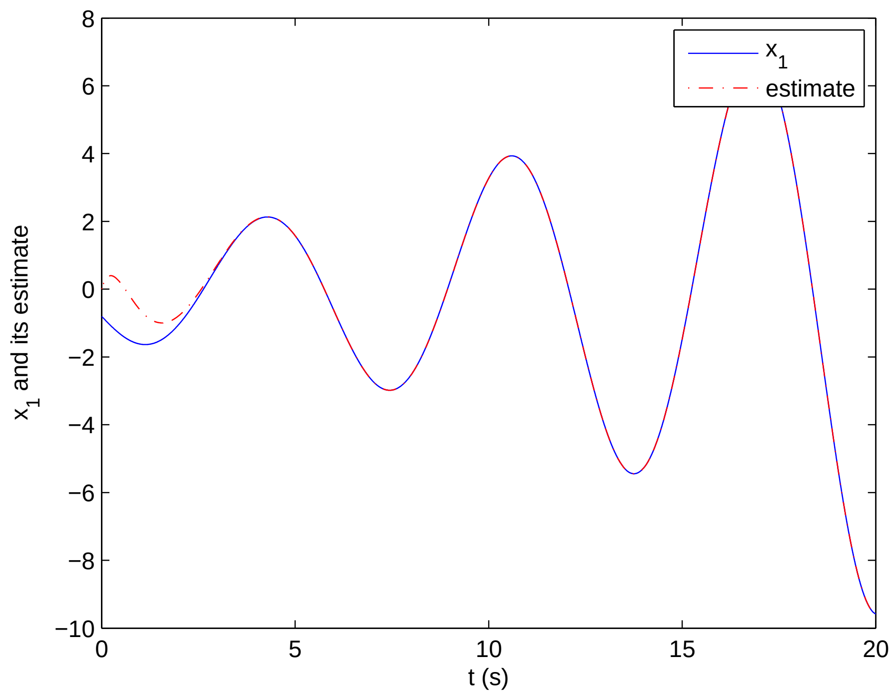

Fig. 6. x_1 and its estimate with initial conditions $x_1(0) = -1$ and $\hat{x}_1(0) = -\hat{z}_1(0) - a\hat{z}_2(0) = 0$.

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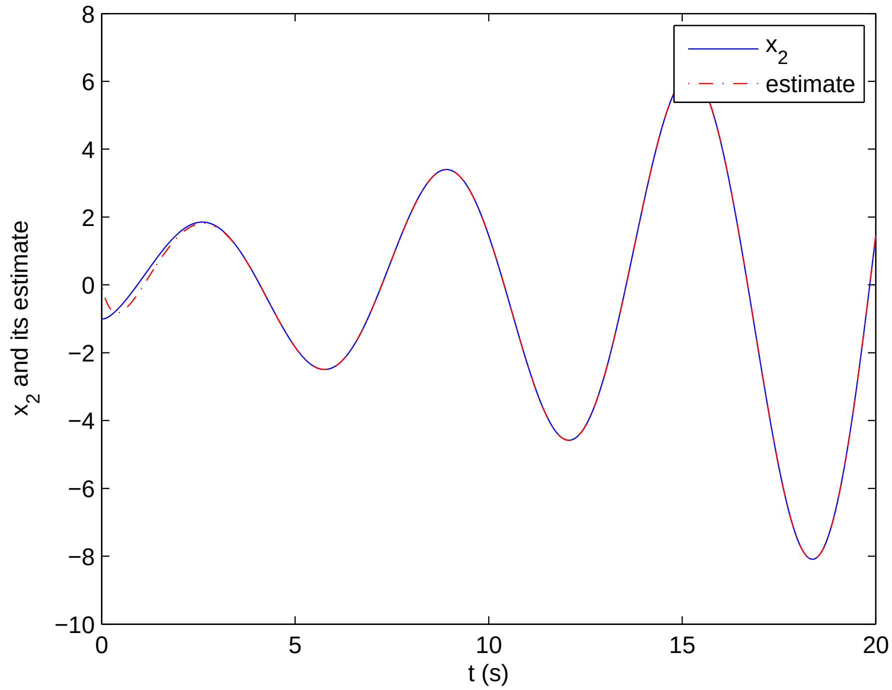

Fig. 7. x_2 and its estimate with initial conditions $x_2(0) = -1$ and $\hat{x}_2(0) = \hat{z}_2(0) = 0$.

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